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ANALYSIS OF THE ONE-DIMENSIONAL MOISTURE MIGRATION CAUSED BY TEMPERATURE GRADIENTS IN A POROUS MEDIUM

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Abstract—An analytical solution is obtained for the one-dimensional moisture migration in a slab of porous material under the influence of temperature gradients. The solution is relevant to such problems as the moisture redistribution in soil under the influence of solar heat, the moisture migration in the vicinity of earth sheltered structures, the measurement of transport properties of porous materials and others.

NOMENCLATURE

c_s, specific heat of the porous material (wet) [$J\,K^{-1}\,kg^{-1}$];
D, vapor diffusivity in air [$m^2\,s^{-1}$];
f, porosity;
Fo, Fourier number for heat transfer;
Fo_m, Fourier number for mass transfer;
H_v, molar enthalpy of vaporization [$J\,mole^{-1}$];
j_v, vapor flux [$kg\,m^{-2}\,s^{-1}$];
j_l, liquid flux [$kg\,m^{-2}\,s^{-1}$];
j_m, total mass flux [$kg\,m^{-2}\,s^{-1}$];
L, slab thickness [m];
M, molar mass [$kg\,mole^{-1}$];
R, ideal gas constant [$J\,mole^{-1}\,K^{-1}$];
$P, P_1, P_2, Q, Q_1, R, R_1$, polynomials;
p_v, vapor pressure [$N\,m^{-2}$];
q, heat flux [$W\,m^{-2}$];
T, temperature (absolute) [K];
T_1, temperature at boundary 1 [K];
T_2, temperature at boundary 2 [K];
T_i, initial temperature [K];
ΔT, temperature drop across slab [K];
W, moisture content [kg moisture/kg dry soil];
$\Delta \mathbf{W}$, non-dimensional moisture content;
x, linear coordinate [m].

Greek symbols

α, thermal diffusivity [$m^2\,s^{-1}$];
ε, tortuosity factor;
ρ_a, air density [$kg\,m^{-3}$];
ρ_v, vapor density [$kg\,m^{-3}$];
ρ_s, bulk density of the porous material (dry) [$kg\,m^{-3}$];
τ, time [s].

1. INTRODUCTION

THE STUDY of combined heat and moisture migration in porous materials is of interest to various fields of engineering and environmental sciences, for instance to manufacturing processes using drying, to the soil scientist's task of predicting the redistribution of soil moisture as influenced by solar heat incidence, and to the energy and environmental analysis of earth sheltered structures.

In the present paper, the problem of unsteady one-dimensional flow of heat and moisture is treated analytically for two cases: constant temperature boundaries and constant heat flux boundaries. It is postulated that no mass flow occurs through the boundaries of the porous medium. The transport properties occurring in the transport equations are assumed constant. This has the advantage to lead to very general relations describing the temperature and moisture field when these are expressed in non-dimensional parameters. A computer solution with the same restrictions is presented in [1]. There, the relationship between the local saturation pressure of the vapor in the pores of the medium and the local temperature was approximated as linear. This approximation was dropped in the present analysis and the influence of the non-linear relation is investigated.

An unsteady method for the measurement of the thermal conductivity of soils has been developed and presented in [2]. In this method, a timewise constant heat flux is applied to one surface of a layer of the soil whereas the other surface is kept adiabatic to heat. The question arises how far the results of the measurements with this method are influenced by a rearrangement of the moisture in the layer. The analysis of the second case treated here aims to answer this question.

2. THE EQUATIONS OF TRANSFER FOR HEAT AND MOISTURE FLOW

The one dimensional equations for the simultaneous flow of heat and moisture in a porous material can be found in a number of papers [e.g. 1–6]. These are the energy and moisture conservation equations, as follows:

$$\rho_s c_s \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) \quad (1)$$

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$$\rho_s \frac{\partial W}{\partial \tau} = \frac{\partial}{\partial x}\left[f\varepsilon D \rho_a \frac{\partial}{\partial x}\left(\frac{\rho_v}{\rho_a}\right)\right] + \frac{\partial}{\partial x}\left(K \frac{\partial W}{\partial x}\right). \quad (2)$$

In the case of constant transport coefficients, equation (1) can be solved independently for the temperature distribution then, with the temperature known (as a function of x and τ), equation (2) can be solved. In the derivation of equation (1) it has been assumed that the enthalpy transported by the liquid moisture is negligibly small, which implies an upper limit for the temperature level. For example, in the case of water in soil, the enthalpy of the liquid can be neglected if the temperature remains below 50°C. It has also been assumed that the gas phase of the porous material is saturated with the vapor of the liquid phase and that the total pressure is a constant.

With the assumption of constant transport coefficients equations (1) and (2) can be written respectively as:

$$\frac{\partial T}{\partial \tau} = \alpha \frac{\partial^2 T}{\partial x^2} \quad (3)$$

$$\frac{\partial W}{\partial \tau} = \frac{f\varepsilon DM}{\rho_s R} \frac{\partial}{\partial x}\left[\frac{1}{T} \frac{\mathrm{d}p_v}{\mathrm{d}T} \frac{\partial T}{\partial x}\right] + K \frac{\partial^2 W}{\partial x^2}. \quad (4)$$

By developing the term in the square bracket, equation (4) reads:

$$\frac{\partial W}{\partial \tau} = \frac{f\varepsilon DM}{\rho_s R}\left[\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{1}{T} \frac{\mathrm{d}p_v}{\mathrm{d}T}\right)\left(\frac{\partial T}{\partial x}\right)^2 + \frac{1}{T} \frac{\mathrm{d}p_v}{\mathrm{d}T} \frac{\partial^2 T}{\partial x^2}\right] + K \frac{\partial^2 W}{\partial x^2}. \quad (4a)$$

The expressions of the fluxes of heat and matter will also be needed in the following for the definition of the various boundary conditions. These are as follows:

the heat flux

$$q = -k \frac{\partial T}{\partial x} \quad (5)$$

the vapor flux

$$j_v = -f D \varepsilon \rho_a \frac{\partial}{\partial x}\left(\frac{\rho_v}{\rho_a}\right) = -f D \varepsilon \frac{M}{R} \frac{1}{T} \frac{\mathrm{d}p_v}{\mathrm{d}T} \frac{\partial T}{\partial x} \quad (6)$$

the liquid moisture flux

$$j_l = -\rho_s K \frac{\partial W}{\partial x} \quad (7)$$

the total mass flux

$$j_m = j_v + j_l \quad (8)$$

[see e.g. refs. 1 and 2].

From equations (3) and (4) it can be seen that the time scales for the simultaneous flow of heat and moisture are determined by the reciprocal of the corresponding diffusivities, i.e. by α^{-1} and K^{-1}. For most soils and other porous materials the moisture diffusivity is much smaller than the thermal diffusivity; thus, the time for the development of the temperature profiles is much shorter than the corresponding time for moisture flow. It is thus permissible to employ the large time (asymptotic) temperature profiles in the temperature dependent terms of equation (4a). We shall apply this procedure to obtain analytical solutions for the two cases mentioned.

3. MOISTURE REDISTRIBUTION IN A SLAB WITH CONSTANT TEMPERATURE BOUNDARIES

Solutions presented before [3, 5] describe drying processes and stipulate vapor flow through the boundaries. In this paper, the porous material is considered to be delimited by two plane parallel surfaces, which are maintained at constant temperature and are impermeable to mass flow. Let L be the slab thickness, T_1 and T_2 the constant temperatures at which the two face surfaces are maintained, T_i the initial (uniform) temperature of the slab and W_i the initial (uniform) moisture content. Then, the boundary condition of the heat equation can be written as:

$$\tau = 0, \quad 0 \le x \le L: \quad T(x, 0) = T_i \quad (9)$$

$$\tau > 0, \quad x = 0: \quad T(0, \tau) = T_1 \quad (10)$$

$$x = L: \quad T(L, \tau) = T_2 \quad (11)$$

while those for the moisture transfer equation can be written:

$$\tau = 0, \quad 0 \le x \le L: \quad W(x, 0) = W_i \quad (12)$$

$$\tau > 0, \quad x = 0: \quad j_m = j_v + j_l = 0 \quad (13)$$

$$x = L: \quad j_m = j_v + j_l = 0. \quad (14)$$

If the flux expressions (6) and (7) are employed, the boundary conditions (13) and (14) can be written also as:

$$x = 0: \quad \rho_s K \left.\frac{\partial W}{\partial x}\right|_{x=0} = -f\varepsilon D \frac{M}{R} \frac{1}{T} \frac{\mathrm{d}p_v}{\mathrm{d}T} \left.\frac{\partial T}{\partial x}\right|_{x=0} \quad (15)$$

$$x = L: \quad \rho_s K \left.\frac{\partial W}{\partial x}\right|_{x=L} = -f\varepsilon D \frac{M}{R} \frac{1}{T} \frac{\mathrm{d}p_v}{\mathrm{d}T} \left.\frac{\partial T}{\partial x}\right|_{x=L}. \quad (16)$$

The solution of the heat equation (3) with boundary conditions (9), (10) and (11) is obtained analytically by separation of variables. For large time, the transient terms in the equation thus obtained vanish, and the steady state temperature profile is found to be simply:

$$T_{\text{steady}} = T_1 + \frac{T_2 - T_1}{L} x \equiv T_1 - \frac{\Delta T}{L} x \quad (17)$$

We shall now turn to finding the solution of the moisture flow equation (4a) with boundary conditions (12), (15) and (16). As stated before, the temperature dependent terms shall be evaluated by assuming the temperature profile to be fully developed, i.e. by making use of equation (17). With this, the moisture transfer equation can be written as:

$$\frac{\partial W}{\partial \tau} = \frac{f\varepsilon DM}{\rho_s R}\left[\frac{d}{dT}\left(\frac{1}{T}\frac{dp_v}{dT}\right)\right]\cdot\left(\frac{\Delta T}{L}\right)^2 + K\frac{\partial^2 W}{\partial x^2} \quad (18)$$

with the boundary conditions:

$$x = 0: \quad \left.\frac{\partial W}{\partial x}\right|_{x=0} = \frac{f\varepsilon DM}{\rho_s R}\left.\left[\frac{1}{T}\frac{dp_v}{dT}\right]\right|_{T=T_1}\cdot\left(\frac{\Delta T}{L}\right) \quad (19)$$

$$x = L: \quad \left.\frac{\partial W}{\partial x}\right|_{x=L} = \frac{f\varepsilon DM}{\rho_s R}\left.\left[\frac{1}{T}\frac{dp_v}{dT}\right]\right|_{T=T_2}\cdot\left(\frac{\Delta T}{L}\right). \quad (20)$$

By making use of the Clausius–Clapeyron equation to express the vapor pressure derivative, equations (18), (19), (20) are respectively:

$$\frac{\partial W}{\partial \tau} = B\left[\frac{d}{dT}\phi(T)\right] + K\frac{\partial^2 W}{\partial x^2} \quad (21)$$

$$\left.\frac{\partial W}{\partial x}\right|_{x=0} = B\left(\frac{L}{K\cdot\Delta T}\right)\phi(T_1) \quad (22)$$

$$\left.\frac{\partial W}{\partial x}\right|_{x=L} = B\left(\frac{L}{K\cdot\Delta T}\right)\phi(T_2) \quad (23)$$

where we have denoted by B the following combination of parameters

$$B \equiv \frac{f\varepsilon DM(\Delta T)^2}{R\rho_s L^2} \quad (24)$$

and by $\phi(T)$ a function of temperature defined as:

$$\phi(T) \equiv \frac{H_v p_v}{RT^3}. \quad (25)$$

We shall approximate the function $\phi(T)$ by a polynomial in T. The lowest order polynomial which maintains the proper character of the temperature dependence is a parabola, thus we shall take:

$$\phi(T) = aT^2 + bT + c; \quad \frac{d\phi}{dT} = 2aT + b \quad (26)$$

where a, b and c are coefficients to be found by curve fitting. By employing the linear temperature profile (17) in conjunction with equation (26), ϕ and its derivative with respect to temperature can be replaced by two functions of x, ϕ and ϕ_d as follows:

$$\phi(T) \equiv \phi(x) = \alpha_1 x^2 + \beta_1 x + \gamma_1;$$
$$\phi'(T) \equiv \phi_d(x) = \alpha x + \beta \quad (27)$$

where α_1, β_1, γ_1, α and β, are coefficients which can be expressed in terms of a, b and c by the relations:

$$\alpha_1 = a\cdot\left(\frac{\Delta T}{L}\right)^2; \quad \beta_1 = -(2aT_1 + b)\cdot\left(\frac{\Delta T}{L}\right)$$
$$\gamma_1 = aT_1^2 + bT_1 + c \quad (28)$$
$$\alpha = -2a\cdot\left(\frac{\Delta T}{L}\right); \quad \beta = b + 2aT_1.$$

Equation (21) can be brought to the form of the heat conduction equation by the following change of the dependent variables:

$$W(x,\tau) = \theta(x,\tau) - \frac{B}{K}\psi(x) \quad (29)$$

where θ is the new dependent variable and ψ is a function of x such that:

$$\psi''(x) = \phi_d(x). \quad (30)$$

From the definition (30), the form of the function $\psi(x)$ can be found as follows. Considering equations (26) and (27) we can write:

$$\psi'(x) = \int_0^x \phi_d(x)dx = -\left(\frac{L}{\Delta T}\right)\cdot[\phi(T) - \phi(T_1)] \quad (31)$$

and thus:

$$\psi'(L) = -\left(\frac{L}{\Delta T}\right)\cdot[\phi(T_2) - \phi(T_1)]$$
$$\text{and } \psi'(0) = 0. \quad (32)$$

Also from equations (30) and (31) and the second of the equations (27) results:

$$\psi'(x) = \int_0^x (\alpha x + \beta)dx = \frac{\alpha}{2}x^2 + \beta x \quad (33)$$

and

$$\psi(x) = \frac{\alpha}{6}x^3 + \frac{\beta}{2}x^2. \quad (34)$$

By taking into account equation (32), it can be easily verified that with the change of variables (31) the moisture transfer equation (21) and the boundary conditions (12), (22) and (23) become respectively:

$$\frac{\partial \theta}{\partial \tau} = K\frac{\partial^2 \theta}{\partial x^2} \quad (35)$$

$$\tau = 0: \quad \theta(x,0) = W_i + \frac{B}{K}\psi(x) \quad (36)$$

$$\tau > 0: \quad \left.\frac{\partial \theta}{\partial x}\right|_{x=0} = B\left(\frac{L}{K\cdot\Delta T}\right)\phi(T_1) \equiv -\frac{Q}{K}$$
$$\left.\frac{\partial \phi}{\partial x}\right|_{x=L} = B\left(\frac{L}{K\cdot\Delta T}\right)\phi(T_1) \equiv -\frac{Q}{K} \quad (37)$$

where we have denoted:

$$Q \equiv -B\left(\frac{L}{\Delta T}\right)\phi(T_1). \quad (38)$$

It can be seen that the moisture transfer equation has been reduced to the form of the heat conduction equation, and that the moisture boundary conditions correspond to a constant boundary heat flux. The solution of the θ-equation can thus be derived from the solution of the heat conduction equation with constant boundary heat flux which is given later in this paper. We anticipate, by giving here the solution for the present case, with equal flux at both boundaries:

$$\theta(x,\tau) = \frac{LQ}{K}\Bigg[\Bigg(P\left(\frac{x}{L}\right) - \sum_{n=1}^{\infty}\frac{2}{(\pi n)^2} \times e^{-(\pi n)^2 K\tau/L^2}\cos \pi n\frac{x}{L}\Bigg) - \Bigg(P_1\left(\frac{x}{L}\right) - \sum_{n=1}^{\infty}\frac{2\cos \pi n}{(\pi n)^2} e^{-(\pi n)^2 K\tau/L^2}\cos \pi\frac{x}{L}\Bigg)\Bigg] + \beta\frac{BL^2}{K}\sum_{n=1}^{\infty}\frac{2\cos \pi n}{(\pi n)^2} \times e^{-(\pi n)^2 K\tau/L^2}\cos \pi\frac{x}{L} + \alpha\frac{BL^3}{K} \times \sum_{n=1}^{\infty}\left(\frac{\cos \pi n}{(\pi n)^2} + \frac{2(1-\cos \pi n)}{(\pi n)^4}\right) \times e^{-(\pi n)^2 K\tau/L^2}\cos \pi\frac{x}{L} + \frac{1}{6}\beta\frac{BL^2}{K} + \frac{1}{24}\alpha\frac{BL^3}{K} + W_i \quad (39)$$

where series of polynomials have been denoted as follows:

$$P\left(\frac{x}{L}\right) = \frac{1}{2}\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right) + \frac{1}{3} = \sum_{n=1}^{\infty}\frac{2}{(\pi n)^2}\cos \pi n\frac{x}{L}$$

$$P_1\left(\frac{x}{L}\right) = \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{6} = \sum_{n=1}^{\infty}\frac{2\cos \pi n}{(\pi n)^2}\cos \pi\frac{x}{L}$$

$$P_2\left(\frac{x}{L}\right) = \frac{1}{6}\left(\frac{x}{L}\right)^3 - \frac{1}{24} = \sum_{n=1}^{\infty}\left(\frac{\cos \pi n}{(\pi n)^2} + \frac{2(1-\cos \pi n)}{(\pi n)^4}\right)\cos \pi n\frac{x}{L}. \quad (40)$$

By reverting to the functional transformation (29), the solution can be written in terms of the original variable $W(x,\tau)$, as follows:

$$W(x,\tau) = \frac{LQ}{K}\Bigg[P\left(\frac{x}{L}\right) - \sum_{n=1}^{\infty}\frac{2}{(\pi n)^2} \times e^{-(\pi n)^2 K\tau/L^2}\cos \pi n\frac{x}{L} - P_1\left(\frac{x}{L}\right) + \sum_{n=1}^{\infty}\frac{2\cos \pi n}{(\pi n)^2} e^{-(\pi n)^2 K\tau/L^2}\cos \pi n\frac{x}{L}\Bigg] + \beta\frac{BL^2}{K}\sum_{n=1}^{\infty}\frac{2\cos \pi n}{(\pi n)^2} \times e^{-(\pi n)^2 K\tau/L^2}\cos \pi n\frac{x}{L} + \alpha\frac{BL^3}{K} \times \sum_{n=1}^{\infty}\left(\frac{\cos \pi n}{(\pi n)^2} + \frac{2(1-\cos \pi n)}{(\pi n)^4}\right) \times e^{-(\pi n)^2 K\tau/L^2}\cos \pi n\frac{x}{L} - \beta\frac{BL^2}{K}P_1\left(\frac{x}{L}\right) - \alpha\frac{BL^3}{K}P_2\left(\frac{x}{L}\right) + W_1. \quad (41)$$

On the right hand side of equations (40), the Fourier expansion of the polynomials is given, with which the solution (41) can be verified to satisfy the initial condition.

It is useful to write the solution (41) in non-dimensional form. For this, we introduce the following non-dimensional parameters:

$$D_t \equiv \frac{f\varepsilon DMH_v p_v}{R^2 \rho_s T_1^3} \quad (42)$$

$$\Delta\mathbf{W} = \frac{W - W_i}{\left[\frac{D_t}{K}\cdot \Delta T\right]} \quad (43)$$

$$Fo_m = \frac{K\tau}{L^2} \quad (44)$$

$$\chi_1 = \frac{1}{\phi}\frac{d\phi}{dT}\bigg|_{T=T_1}\cdot \Delta T \cong \left[\frac{H_v}{RT_1} - 3\right]\cdot\left(\frac{\Delta T}{T_1}\right) \quad (45)$$

$$\chi_2 = \frac{1}{\phi}\frac{d}{dT}\left(\frac{d\phi}{dT}\right)\bigg|_{T=T_1}\cdot(\Delta T)^2 \cong \left[\left(\frac{H_v}{RT_1}\right)^2 - 8\left(\frac{H_v}{RT_1}\right) + 12\right]\cdot\left(\frac{\Delta T}{T_1}\right)^2. \quad (46)$$

With these, the solution (41) can be written in non-dimensional form, as follows:

$$\Delta\mathbf{W} = \left[P_1\left(\frac{x}{L}\right) - \sum_{n=1}^{\infty}\frac{2\cos \pi n}{(\pi n)^2} \times e^{-(\pi n)^2 Fo_m}\cos \pi n\frac{x}{L}\right] - \left[P\left(\frac{x}{L}\right) - \sum_{n=1}^{\infty}\frac{2}{(\pi n)^2} \times e^{-(\pi n)^2 Fo_m}\cos \pi n\frac{x}{L}\right] - \chi_1\left[P_1\left(\frac{x}{L}\right) - \sum_{n=1}^{\infty}\frac{2\cos \pi n}{(\pi n)^2} \times e^{-(\pi n)^2 Fo_m}\cos \pi n\frac{x}{L}\right] + \chi_2\left[P_2\left(\frac{x}{L}\right) - \sum_{n=1}^{\infty}\left(\frac{\cos \pi n}{(\pi n)^2} + \frac{2(1-\cos \pi n)}{(\pi n)^4}\right) \times e^{-(\pi n)^2 Fo_m}\cos \pi n\frac{x}{L}\right]. \quad (47)$$

It is interesting to note that the parameters D_t, χ_1 and χ_2 are evaluated at the warm face temperature, T_1. The non-dimensional moisture content, $\Delta\mathbf{W}$, is a negative quantity. The two parameters χ_1 and χ_2 could be replaced at will by their expressions in terms of the physically more meaningful quantities H_v/RT_1 and $\Delta T/T_1$. If ΔT is small the parameters χ_1 and χ_2 are small and the terms containing them can be neglected.

Thus, for the case of small temperature difference between the two faces the solution can be written as:

$$\Delta W = \left[P_1\left(\frac{x}{L}\right) - \sum_{n=1}^{\infty} \frac{2\cos \pi n}{(\pi n)^2} \times e^{-(\pi n)^2 Fo_m} \cos \pi n \frac{x}{L}\right] - \left[P\left(\frac{x}{L}\right) - \sum_{n=1}^{\infty} \frac{2}{(\pi n)^2} \times e^{-(\pi n)^2 Fo_m} \cos \pi n \frac{x}{L}\right]. \quad (48)$$

From the above, it can be seen that in the general case the solution is of the form:

$$\Delta W = f\left(\frac{x}{L}, Fo_m, \chi_1, \chi_2\right)$$

$$\text{or } \Delta W = f\left(\frac{x}{L}, Fo_m, \frac{H_v}{RT_1}, \frac{\Delta T}{T_1}\right)$$

while for small ΔT this can be simplified to:

$$\Delta W = f\left(\frac{x}{L}, Fo_m\right).$$

The solution represented by equation (47) has been plotted in Figs. 1, 2 and 3 for different values of the

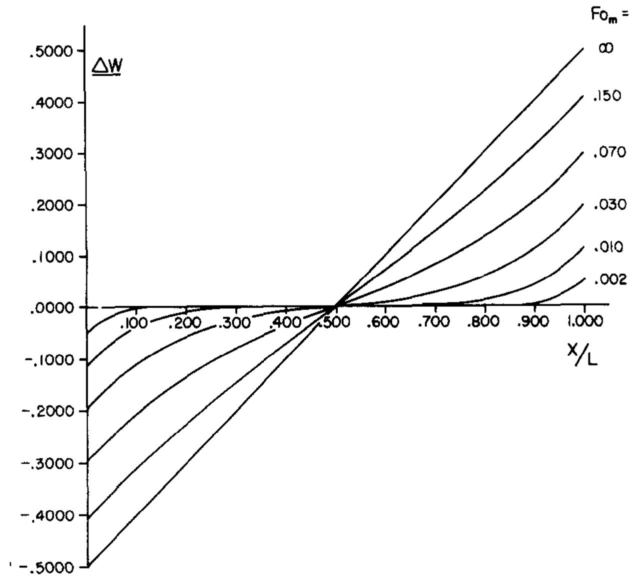

FIG. 1. $\Delta T/T_1 = 0.0003$.

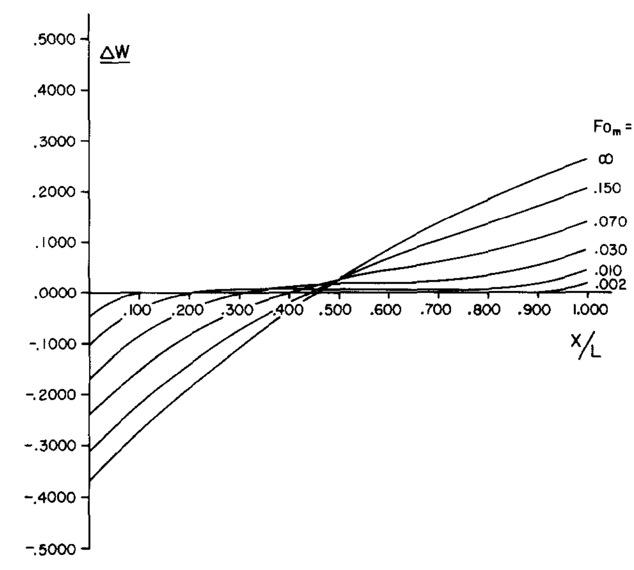

FIG. 2. $\Delta T/T_1 = 0.0660$.

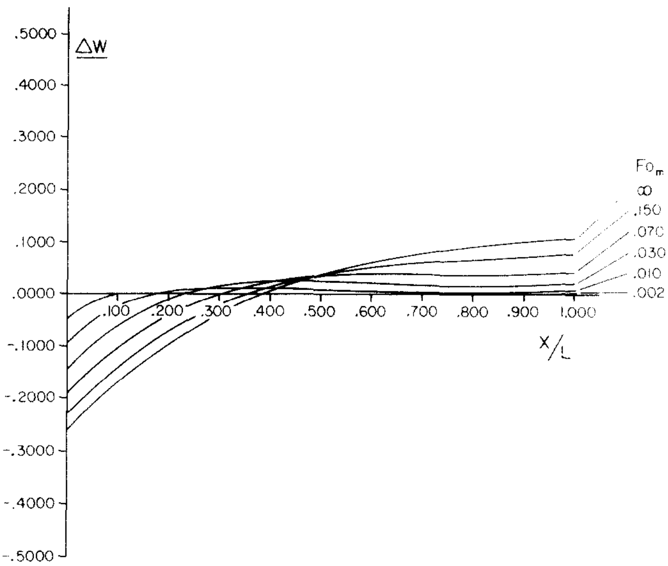

FIG. 3. $\Delta T/T_1 = 0.1548$.

FIGS. 1–3. Rearrangement of the non-dimensional moisture content in a porous slab with mass transfer Fourier number as parameter. The temperature at the left surface has suddenly been raised whereas the temperature of the right hand surface is kept at its original value.

non-dimensional parameters. As seen from Fig. 1, in which $H_v/RT_1 = 16.0$ and $\Delta T/T_1 = 0.0003$, for a small temperature difference between the two faces the moisture profiles are very nearly symmetric about the slab mid-plane. In Fig. 2, $H_v/RT_1 = 17.4$ and $\Delta T/T_1 = 0.066$. In Fig. 3, $H_v/RT_1 = 16.0$ and $\Delta T/T_1 = 0.155$, showing that for large ΔT the profiles are skewed, which is an effect of the non-linear dependence of the vapor pressure on temperature. For large time the transient terms in equation (47) vanish and the steady state solution is found to be:

$$(\mathbf{\Delta W})_\infty = \frac{x}{L} - \frac{1}{2} - \chi_1\left[\frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{6}\right] + \chi_2\left[\frac{1}{6}\left(\frac{x}{L}\right)^3 - \frac{1}{24}\right] \quad (49)$$

while for small χ_1 and χ_2 (small ΔT) the steady state solution becomes simply:

$$(\mathbf{\Delta W})_\infty = \frac{x}{L} - \frac{1}{2}. \quad (50)$$

Numerical example

Equations (47)–(50) can be applied to predict the moisture redistribution under the influence of temperature gradients in actual cases, if the properties of the porous material are known.

As an example, consider the case of a concrete slab of thickness $L = 0.25$ m, maintained at a temperature difference $\Delta T = 20$ K between the two faces and a warm face temperature $T_1 = 303$ K. If the initial, uniform moisture content (water) is $W_i = 0.05$, it is required to determine: (a) the moisture content at the warm face at time $\tau = 100$ h; (b) the steady state (long time) moisture content at the warm face location. The various physical properties are assumed to be: porosity, $f = 0.35$; vapor diffusivity in the porous medium, $\varepsilon D = 2.07 \times 10^{-5}\ \mathrm{m^2\ s^{-1}}$; dry density, $\rho_s = 1670\ \mathrm{kg\ m^{-3}}$; liquid moisture diffusivity, $K = 5.0 \times 10^{-9}\ \mathrm{m^2\ s^{-1}}$; molar enthalpy of vaporization $H_v = 4.37 \times 10^4\ \mathrm{J\ mole^{-1}}$; vapor pressure at warm face temperature, $p_v = 4.24 \times 10^3\ \mathrm{N\ m^{-2}}$.

First, the various non-dimensional parameters have to be computed. These are found to be: $D_t = 7.532 \times 10^{-12}$; $H_v/RT_1 = 17.36$; $\Delta T/T_1 = 6.6 \times 10^{-2}$; $\chi_1 = 0.948$; $\chi_2 = 0.760$. It can be seen that this case corresponds to the solution displayed in Fig. 2.

(a) $Fo_m = K\tau/L^2 = 2.88 \times 10^{-2} \cong 3.0 \times 10^{-2}$, and from Fig. 2 it is found $(\mathbf{\Delta W})|_{x=0} = -0.17$. Then, from equation (43), $W|_{x=0} = 0.045$.
(b) From equation (49) or from Fig. 2:

$$(\mathbf{\Delta W})_{\text{steady}}|_{x=0} = -\frac{1}{2} + \frac{1}{6}\chi_1 - \frac{1}{24}\chi_2 = -0.37$$

and thus

$$W_{\text{steady}}|_{x=0} = 0.039.$$

An estimate of the time at which the steady state is nearly reached can be effected by taking $Fo_m = 1$, and thus, $\tau \cong L^2/K = 1.25 \times 10^7$ s or 145 days.

4. HEAT AND MOISTURE TRANSFER WITH CONSTANT HEAT FLUX BOUNDARIES

The porous material is considered to be delimited by two plane parallel surfaces, which are impermeable to mass flow and on which constant heat fluxes are applied. The boundary conditions of the heat equation can be written as:

$$\tau = 0, \quad 0 \le x \le L: \quad T(x,0) = T_i \quad (51)$$

$$\tau > 0, \qquad x = 0: \quad -k\frac{\partial T}{\partial x}\bigg|_{x=0} = q_1 \quad (52)$$

$$x = L: \quad -k\frac{\partial T}{\partial x}\bigg|_{x=L} = q_2 \quad (53)$$

while those for the moisture transfer equation are given by equations (12), (13) and (14).

The solution of equation (3) with boundary conditions (51), (52) and (53) is obtained by separation of variables. The result is:

$$T(x,\tau) = T_i + \frac{Lq_1}{k}\left[\frac{\alpha\tau}{L^2} + P\left(\frac{x}{L}\right) - \sum_{n=1}^{\infty}\frac{2}{(\pi n)^2}e^{-(\pi n)^2\cdot Fo}\cos \pi n\frac{x}{L}\right] - \frac{Lq_2}{k}\left[\frac{\alpha\tau}{L^2} + P_1\left(\frac{x}{L}\right) - \sum_{n=1}^{\infty}\frac{2\cos \pi n}{(\pi n)^2}e^{-(\pi n)^2\cdot Fo}\cos n\frac{x}{L}\right]. \quad (54)$$

The quasisteady (large time) temperature profile is obtained by dropping the transient terms in equation (54). It thus becomes:

$$T_{q\cdot \text{steady}} = T_i + \frac{q_1 - q_2}{L\rho_s c_s}\tau + \frac{Lq_1}{k}P\left(\frac{x}{L}\right) - \frac{Lq_2}{k}P_1\left(\frac{x}{L}\right) \quad (55)$$

where $P(x/L)$ and $P_1(x/L)$ are the polynomials defined by equations (40).

As was noted before, the temperature profiles develop much faster than the corresponding moisture profiles. This justifies the use of the quasi-steady temperature distribution (55) for the evaluation of the temperature dependent terms in the moisture transfer equation (4). We shall introduce an important simplification which applies in the case of small slab thickness. For this, we shall prove that if the slab thickness is sufficiently small the first term in the square brackets of equation (4a) can be neglected against the second term.

The two terms can be evaluated by the following approximate relationships:

$$\frac{f\varepsilon DM}{\rho_s R}\cdot\frac{1}{T}\cdot\frac{dp_v}{dT}\cdot\frac{\partial^2 T}{\partial x^2} \cong \frac{f\varepsilon DM}{\rho_s R}\cdot\frac{1}{T}\frac{dp_v}{dT}\cdot\frac{q}{Lk} \quad (56)$$

$$\frac{f\varepsilon DM}{\rho_s R}\cdot\frac{d}{dT}\left(\frac{1}{T}\frac{dp_v}{dT}\right)\cdot\left(\frac{\partial T}{\partial x}\right)^2 \cong \frac{f\varepsilon DM}{\rho_s R}\cdot\frac{d}{dT}\left(\frac{1}{T}\frac{dp_v}{dT}\right)\cdot\frac{q^2}{k^2} \quad (57)$$

where q is the heat flux applied at the boundary $x = 0$. The term containing the second derivative $\partial^2 T/\partial x^2$ is much smaller than the term in $(\partial T/\partial x)^2$ if:

$$\frac{1}{T}\cdot\frac{dp_v}{dT}\cdot\frac{q}{Lk} \gg \frac{d}{dT}\left(\frac{1}{T}\frac{dp_v}{dT}\right)\cdot\frac{q^2}{k^2} \quad (58)$$

or

$$\left(\frac{Lq}{k}\right) \ll \frac{T}{\left[\dfrac{H_v}{RT} - 3\right]}.$$

For a numerical estimate let's take $L = 0.02$ m, $q = 100 \cdot \text{W m}^{-2}$, $k = 1.25\ \text{W m}^{-1}\,\text{K}^{-1}$, $T = 303$ K, $H_v = 4.3 \times 10^4\ \text{J mole}^{-1}$. With these it is found:

$$\frac{Lq}{k} = 1.60 \quad \text{and} \quad \frac{T}{\left[\dfrac{H_v}{RT} - 3\right]} = 21.50$$

and thus, the condition (58) can be considered to be well fulfilled. Since the above numerical values are typical for experimental methods of soil thermal conductivity measurement (see [2]), we can consider the condition (58) to be satisfied for this very situation. If condition (58) is satisfied, the moisture transfer equation (4) reduces to:

$$\frac{\partial W}{\partial \tau} = \frac{f\varepsilon DM(q_1 - q_2)}{\rho_s RkL}\cdot\phi(T) + K\frac{\partial^2 W}{\partial x^2} \quad (59)$$

where the notation (26) has been employed.

The moisture transfer boundary condition (13) and (14) can be written after due transformations as:

$$x = 0: \quad \frac{\partial W}{\partial x}\bigg|_{x=0} = \frac{f\varepsilon DMq_1}{\rho_s RkL}\cdot\frac{L}{K}\cdot\phi(T)\bigg|_{x=0} \quad (60)$$

$$x = L: \quad \frac{\partial W}{\partial x}\bigg|_{x=L} = \frac{f\varepsilon DMq_2}{\rho_s RkL}\cdot\frac{L}{K}\cdot\phi(T)\bigg|_{x=L}. \quad (61)$$

By denoting:

$$S_1 = \frac{f\varepsilon DMq_1}{\rho_s RkL} \quad \text{and} \quad S_2 = \frac{f\varepsilon DMq_2}{\rho_s RkL} \quad (62)$$

the moisture transfer equation and its boundary conditions are transformed into:

$$\frac{\partial W}{\partial \tau} = (S_1 - S_2)\phi(T) + K\frac{\partial^2 W}{\partial x^2} \quad (63)$$

$$\tau = 0, \quad W(x,0) = W_i \quad (64)$$

$$\tau > 0, \quad x = 0: \quad \frac{\partial W}{\partial x}\bigg|_{x=0} = \frac{S_1 L}{K}\phi(T)\bigg|_{x=0} \quad (65)$$

$$x = L: \quad \frac{\partial W}{\partial x}\bigg|_{x=L} = \frac{S_2 L}{K}\phi(T)\bigg|_{x=L}. \quad (66)$$

Since the slab thickness is small, as imposed by condition (58), we shall consider the function $\phi(T)$ to be evaluated at the mid-plane location and thus to be a function of time but not a function of x also. This is so because, if the slab thickness is small, the temperature difference between the two faces is small and the variation of T with x can be ignored in the evaluation of $\phi(T)$. The temperature at the location of the mid-plane is:

$$T \cong T_{av} = T_i + \frac{q_1 - q_2}{L\rho_s c_s}\tau - \frac{1}{24}\frac{(q_1 - q_2)L}{k}. \quad (67)$$

With this, $\phi(T)$ can be seen as the following function of τ:

$$\phi(T) \equiv \phi(\tau) = \varepsilon\tau^2 + \delta\tau + \gamma \quad (68)$$

where the coefficients ε, δ, γ can be expressed in terms of the coefficients a, b and c introduced before (see equation (26)). If we denote:

$$A = T_i - \frac{1}{24}\cdot\frac{q_1 - q_2}{k}\cdot L; \quad B = \frac{q_1 - q_2}{L\rho_s c_s} \quad (69)$$

the relationships expressing ε, δ and γ in terms of a, b and c are:

$$\varepsilon = aB^2; \ \delta = 2aAB + bB; \ \gamma = aA^2 + bA + c. \quad (70)$$

The moisture transfer equation (63) can be reduced to the form of the heat conduction equation by the following change of the dependent variable:

$$W(x,\tau) = \theta(x,\tau) + (S_1 - S_2) \times \left(\frac{\varepsilon}{3}\tau^3 + \frac{\delta}{2}\tau^2 + \gamma\tau\right). \quad (71)$$

With this, the moisture equation and the boundary conditions become:

$$\frac{\partial\theta}{\partial\tau} = K\frac{\partial^2\theta}{\partial x^2}$$

$$\tau = 0, \ \theta(x,0) = W_i$$

$$\tau > 0, \ x = 0: \ \left.\frac{\partial\theta}{\partial x}\right|_{x=0} = \frac{S_1 L}{K}\phi(\tau) = -\frac{Q_1(\tau)}{K} \quad (72)$$

$$x = L: \ \left.\frac{\partial\theta}{\partial x}\right|_{x=L} = \frac{S_2 L}{K}\phi(\tau) = -\frac{Q_2(\tau)}{K} \quad (73)$$

where we have denoted:

$$Q_1(\tau) = -S_1 L\phi(\tau); \ Q_2(\tau) = -S_2 L\phi(\tau). \quad (74)$$

It can be seen that the problem of solving the θ-equation is similar to that of solving the heat conduction equation with given boundary heat fluxes. The solution (54) could be applied directly if the equivalent fluxes Q_1 and Q_2 were not functions of time. As it is, we must apply Duhamel's theorem (see [7]) to derive the solution of the θ-equation from the solution of constant heat flux boundaries. This has been done here byt the complete derivations shall not be given. We shall give only the final result in terms of the original variable $W(x,\tau)$:

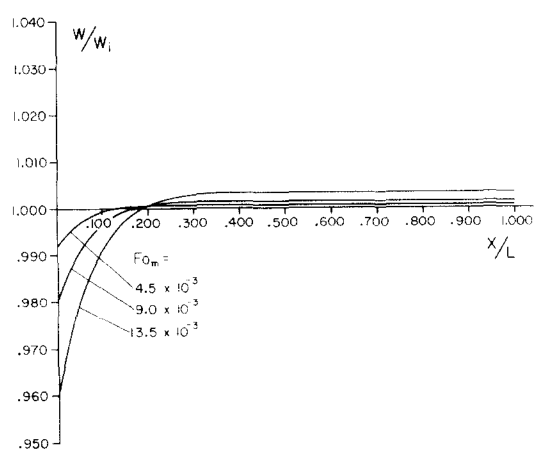

FIG. 4. Early stages of the rearrangement of the moisture ratio in a porous slab with mass transfer Fourier number as parameter. A constant heat flux is being applied to the left surface, whereas the right surface is kept adiabatic to heat flow.

$$W(x,\tau) = W_i - S_1\frac{L^2}{K}(\varepsilon\tau^2 + \delta\tau + \gamma)\cdot P\left(\frac{x}{L}\right)$$
$$- S_1\frac{L^4}{K_2}(2\varepsilon\tau + \delta)\cdot Q\left(\frac{x}{L}\right)$$
$$- S_1\frac{L^6}{K_3}(2\varepsilon)\cdot R\left(\frac{x}{L}\right)$$
$$+ S_2\frac{L^2}{K_2}(\varepsilon\tau^2 + \delta\tau + \gamma)\cdot P_1\left(\frac{x}{L}\right)$$
$$+ S_2\frac{L^4}{K_2}(2\varepsilon\tau + \delta)\cdot Q_1\left(\frac{x}{L}\right)$$
$$+ S_2\frac{L^6}{K^3}\cdot(2\varepsilon)\cdot R_1\left(\frac{x}{L}\right)$$
$$+ S_1\frac{L^6}{K^3}\cdot(2\varepsilon)\cdot\sum_{n=1}^{\infty}\frac{2}{(\pi n)^6}\cdot e^{-(\pi n)^2 k\tau/L^2}\cos\pi n\frac{x}{L}$$
$$- S_1\frac{L^4}{K^2}\cdot(\delta)\cdot\sum_{n=1}^{\infty}\frac{2}{(\pi n)^4}e^{-(\pi n)^2 k\tau/L^2}\cos\pi n\frac{x}{L}$$
$$+ S_1\frac{L^2}{K}\cdot(\gamma)\cdot\sum_{n=1}^{\infty}\frac{2}{(\pi n)^2}\cdot e^{-(\pi n)^2 k\tau/L^2}\cos\pi n\frac{x}{L}$$
$$- S_2\frac{L^6}{K^3}\cdot(2\varepsilon)\cdot\sum_{n=1}^{\infty}\frac{2\cos\pi n}{(\pi n)^6}\cdot e^{-(\pi n)^2 k\tau/L^2}\cos\pi n\frac{x}{L}$$
$$+ S_2\frac{L^4}{K^2}\cdot(\delta)\cdot\sum_{n=1}^{\infty}\frac{2\cos\pi n}{(\pi n)^4}\cdot e^{-(\pi n)^2 k\tau/L^2}\cos\pi n\frac{x}{L}$$
$$- S_2\frac{L^2}{K}\cdot(\gamma)\cdot\sum_{n=1}^{\infty}\frac{2\cos\pi n}{(\pi n)^2}\cdot e^{-(\pi n)^2 k\tau/L^2}\cos\pi n\frac{x}{L}. \tag{75}$$

In equation (75), Q, R, Q_1 and R_1 are the following simple polynomyals:

$$Q\left(\frac{x}{L}\right) = \frac{1}{24}\left(\frac{x}{L}\right)^4 - \frac{1}{6}\left(\frac{x}{L}\right)^3 + \frac{1}{6}\left(\frac{x}{L}\right)^2 - \frac{1}{45} = -\sum_{n=1}^{\infty}\frac{2}{(\pi n)^4}\cos\pi n\frac{x}{L}$$
$$R\left(\frac{x}{L}\right) = \frac{1}{720}\left(\frac{x}{L}\right)^6 - \frac{1}{120}\left(\frac{x}{L}\right)^5 + \frac{1}{72}\left(\frac{x}{L}\right)^4 - \frac{1}{90}\left(\frac{x}{L}\right)^2 + \frac{2}{945} = \sum_{n=1}^{\infty}\frac{2}{(\pi n)^6}\cos\pi n\frac{x}{L}$$
$$Q_1\left(\frac{x}{L}\right) = \frac{1}{24}\left(\frac{x}{L}\right)^4 - \frac{1}{12}\left(\frac{x}{L}\right)^2 + \frac{7}{360} = -\sum_{n=1}^{\infty}\frac{2\cos\pi n}{(\pi n)^4}\cos\pi n\frac{x}{L}$$
$$R_1\left(\frac{x}{L}\right) = \frac{1}{720}\left(\frac{x}{L}\right)^6 - \frac{1}{144}\left(\frac{x}{L}\right)^4 + \frac{7}{720}\left(\frac{x}{L}\right)^2 - \frac{31}{15120} = \sum_{n=1}^{\infty}\frac{2\cos\pi n}{(\pi n)^6}\cos\pi n\frac{x}{L}. \tag{76}$$

In the right hand side of equations (76), the Fourier expansions of the polynomials are given with which the solution (75) can be verified to satisfy the initial condition.

The solution (75) is presented in Fig. 4, for one case of actual measurement of the soil thermal conductivity. The values of the various parameters employed in the calculation are as follows: slab thickness, L = 0.02 m, heat flux at face 1: $q_1 = q = 206$ W m^{-2}, heat flux at face 2: $q_2 = 0$, initial uniform temperature: T_i = 278 K, initial uniform moisture content: $W_i = 0.05$, soil porosity: $f = 0.35$, dry soil density: $\rho_s = 1670$ kg m^{-3}, wet soil specific heat: $c_s = 1000$ J K^{-1} kg^{-1}, soil thermal conductivity: $k = 1.6$ W K^{-1} m^{-1}, vapor diffusivity in soil: $\varepsilon D = 2.07 \times 10^{-5}$ m^2 s^{-1}, liquid moisture (water) diffusivity: $K = 1.0 \times 10^{-9}$ m^2 s^{-1}, molar enthalpy of vaporization: $H_v = 4.4 \times 10^4$ J mole^{-1}.

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ANALYSE DE LA MIGRATION MONODIMENSIONNELLE DE L'HUMIDITE CAUSEE PAR DES GRADIENTS DE TEMPERATURE DANS UN MILIEU POREUX

Résumé—On obtient une solution analytique pour la migration monodimensionnelle d'humidité dans une plaque poreuse sous l'influence des gradients de température. La solution est applicable aux problèmes de la redistribution d'humidité dans le sol sous l'effet du rayonnement solaire, de la migration d'humidité au voisinage des structures du manteau terrestre, de la mesure des propriétés de transport des matériaux poreux et d'autres problèmes.

UNTERSUCHUNG DES EINDIMENSIONALEN FEUCHTIGKEITSTRANSPORTS AUFGRUND VON TEMPERATURGEFÄLLEN IN EINEM PORÖSEN MEDIUM

Zusammenfassung—Es wurde eine analytische Lösung für den eindimensionalen Feuchtigkeitstransport in einer Platte aus porösen Material unter dem Einfluß von Temperaturgefällen gefunden. Die Lösung ist wichtig für Probleme wie z. B. die Feuchtigkeitsverteilung im Erdreich unter dem Einfluß der Solarwärme, den Feuchtigkeitstransport in der Nähe von erdbedeckten Strukturen, das Messen der Transportgrößen von porösen Stoffen und anderes mehr.

ОДНОМЕРНЫЙ АНАЛИЗ МИГРАЦИИ ВЛАГИ, ВЫЗВАННОЙ ТЕМПЕРАТУРНЫМИ ГРАДИЕНТАМИ В ПОРИСТОЙ СРЕДЕ

Аннотация—Получено одномерное аналитическое решение задачи о миграции влаги в плите из пористого материала под влиянием температурных градиентов. Решение справедливо для таких задач, как перераспределение влаги в почве под влиянием солнечного тепла, миграции влаги вблизи конструкций, заглубленных в грунт, измерение переносных свойств пористых материалов и т. д.